Radar Detection of Spherical Particles

In this section we derive the RADAR EQUATION which expresses the power returned to a radar by spherical particles of either ice or water. We will first derive a general equation that relates the returned power to target characteristics. Then we will relate these characteristics to the specific cases of spherical water or ice particles. The radar equation allows for the backscattered power to be related to meteorological information.

Define the following terms:

\[ P_t = \text{transmitted power in watts (this would be the average transmit power over the duration of a pulse that is radiated to free space)} \]

\[ A_t = \text{the area of a small target intercepting the transmitted power} \]

\[ r = \text{range to target (distance along the beam in this case)} \]

\[ P_\sigma = \text{power intercepted by the target} \]
Radar Equation-con’t

Assuming isotropic radiation from the transmitter, the radiation power intercepted by the target is simply,

\[ P_\sigma = \frac{P_t A_t}{(4\pi r^2)} \quad (1) \]

Radars are of course not isotropic radiators, instead high gain antennas are employed to focus the EM energy into a narrow beam through the use of a parabolic reflector. The gain G is

\[(\text{power gain}) \; G = \text{power density along the beam/isotropic power density}\]

Therefore, (1) is rewritten to account for gain G as,

\[ P_\sigma = \frac{GP_t A_t}{(4\pi r^2)} \quad (2) \]

Our primary interest is to determine the power returned from the target. The details of the transmit power/target interaction will be discussed later, but for now, let's assume that the power is radiated isotropically by the target. Then the return power is simply,

\[ Pr = \left[ \frac{GP_t A_t}{(4\pi r^2)} \right] \left( \frac{A_e}{(4\pi r^2)} \right) \quad (3) \]

where \( A_e \) is the effective cross-section of the receiving antenna.

Here we have obviously assumed that all the power intercepted by the target is re-radiated.
Radar Equation-con’t

The effective cross section of the receiving antenna is given by,

\[ A_e = \rho A_p \]

where \( A_p \) is the physical area of the antenna and \( \rho \) is the antenna efficiency. From theoretical considerations for a circular, paraboloidal antenna,

\[ A_e = G \lambda^2/(4\pi) \text{ and } G = 8 \pi A_p / 3 \lambda^2 \]

Note that for fixed wavelength (\( \lambda \)), a larger reflector provides more gain. Also, for fixed antenna physical area, more gain is realized at shorter wavelength’s. Combining the above equations yields,

\[ A_e = 2/3 \cdot A_p \]

Substituting expressions for \( A_e \) and \( G \) into Eqn. (3) yields expressions for the received power,

\[ P_r = P_t G^2 \lambda^2 A_r/(4\pi)^3 r^4 = P_t A_p^2 A_r/(9\pi\lambda^2 r^4) \quad (4) \]
Backscattering Cross-Section

Obviously meteorological targets do not radiate isotropically since the relationship between particle size and wavelength determine the relative portions of backscatter, forward (Mie) scatter, etc. Hence we introduce the back-scattering cross-section $\sigma$.

The back scattering cross section is defined as the area intercepting that amount of power, which is scattered isotropically, would return to the receiver and amount of power equal to that actually received. In general, $\sigma > A_t$.

Alternate definition of $\sigma$: the area which, when multiplied by the incident power, gives the total power radiated by an isotropic source which radiates the same power in the backward direction as the actual scatterer. Therefore,

$$\sigma P_i = 4\pi r^2 S$$

where $P_i$ is the power density (W/m$^2$) incident on the target at range $r$ and $S$ is the backscattered power density at the receiving antenna. Using the scattering cross section, we can rewrite Eqn. (4) to yield,

$$P_r = P_t G^2 \sigma / (4\pi)^3 r^4 \quad (5)$$

This equation is general, it applies to any target. Problem before us is to determine $\sigma_i$ for meteorological targets of interest to us. Note (5) contains information on the target, radar and range.
Single vs. Distributed Targets

Eqn. (5) applies to a single target. We however deal with distributed targets, such as raindrops, ice particles, hailstones, insects, etc. A pulse volume, typically 1 km\(^3\) in volume (determined by the pulse length and the beamwidth characteristics of the antenna) may contain \(10^{12}\) particles, each illuminated by the radar pulse. We are interested in determining the average return power, \(\bar{P}_r\), which is found by averaging the returned powers over a series of successive pulses. We will relate this averaging period to the error variance in the desired measurand, as well as other radar and meteorological parameters.

Pulse resolution volume \(V_m = \text{pulse length (m)} \cdot \text{area of pulse determined by antenna beamwidth}\)

For a circular beamwidth, \(V_m = \pi (r\theta/2)^2 \cdot h/2 \quad (6)\)

where \(r = \text{range to target}\)
\(\theta = \text{beamwidth in radians (typically 1 degree)}\)
\(h = \text{pulse length in meters (c}\tau/2)\)
\(\tau = \text{pulse duration (typically 1}\ \mu\text{s)}\)

Why is the pulse length in Eqn. (6) equal to \(h/2\), when the physical length of the pulse is \(c\tau\)?
Consider segments along a radar beam, or range gates. In each gate, the individual pulse volume returns are collected and averaged over some time period to produce the estimate for that gate. Typically, a radar beam is segmented into 512, or 1024 range bins, each with a length of $c\tau/2$.

Since the pulse has to travel out and back to the gate, effective pulse length becomes $\frac{1}{2} (c\tau)$ to ensure that the leading edge of the pulse returns to the radar at the same time as the back edge of the pulse. Data are binned into segments with length $\frac{1}{2} c\tau$. 
Sampling considerations; decorrelation

Since the particles in a sample volume are randomly distributed, the power returned from the volume will change due to the relative movement of the scatterers. The time required for reshuffling is proportional to the radar wavelength, and inversely proportional to the amount of turbulence in the pulse volume. The turbulence level is measured by the Doppler spectral width \(\bar{\sigma}\), which is the second moment of the Gaussian velocity distribution. The details of this will be considered in a later chapter.

\[
\text{Decorrelation time} = \frac{\lambda}{\bar{\sigma}} \quad \text{-------------------0.01 seconds for S-band (for reasonable turbulence levels)}
\]

This decorrelation time is the time required for turbulence to re-arrange the targets such that returned power is completely independent of the previous return power. It is clear that a pulsed radar must acquire samples of return power over a period of time (for each range gate) that is at least equal to the decorrelation time. Each sample is correlated to one another, however.

\[
\text{Sample time} \geq \frac{\lambda}{\bar{\sigma}}
\]
The time over which samples are collected in a range gate is known as the DWELL TIME. 

\[ \text{Dwell time} = M \cdot T_s \]

where \( M \) is the number of samples collected and \( T_s \) is the pulse repetition time. This dwell time must be greater than or equal to the decorrelation time.

Hence,

\[ M \cdot T_s \geq \frac{\lambda}{\bar{\sigma}} \]

Assume \( \lambda = 10 \text{ cm} \)
\( \sigma = 2 \text{ m/s} \)
\( T_s = 1 \text{ millisecond} \)

\[ M \geq 50 \text{ samples} \]

Clearly the antenna rotation rate must be slow enough to allow \( M \) samples to be acquired on each ray of data. We will discuss relationships between the angular spacing of successive rays, the antenna rotation rate, PRF and number of samples in each estimate in a later chapter.
Fig. 6.2  Standard error of $\hat{Z}$ estimates (for SNR $\gg 1$) versus $M$ with normalized spectrum width $\sigma_{vn}$ as a parameter.

Doviak and Zrnic (1993)
Radar Reflectivity

Hence the received power is averaged over the time interval required to acquire M samples

\[
P_r = \left[ P_t G^2 \lambda^2 / (4\pi)^3 r^4 \right] \sum_{i=1}^{M} \sigma_i \quad (7)
\]

Since we are dealing with the a pulse volume, it is easier to consider the backscattering cross section per unit volume. The expression for the average received power, \( P_r \) in this format is,

\[
P_r = \left[ P_t G^2 \lambda \theta \phi \ h / 512\pi^2 r^2 \right] \sum_{\text{vol}} \sigma_i \quad (8)
\]

where the units on the summation is area/volume, commonly expressed as cm\(^2\)/m\(^3\). In this equation, \( \theta \) and \( \phi \) represent the horizontal and vertical beamwidths (in radians), respectively.

The summation of the backscattering cross sections per unit volume (\( \sum \sigma_i \)) is defined as the RADAR REFLECTIVITY and represented by the symbol \( \eta \). Units are cm\(^2\)/m\(^3\), which can be interpreted as the total back scattering area per unit volume.
Probert-Jones (1962) Correction

When Eqn. (8) was tested experimentally (using known values of the back scattering cross section and measuring and calculating the mean power), it was found that the calculated value was consistently too large. A corrected form of Eqn. (8) was then derived as,

\[ P_r = \left[ P_t G^2 \lambda \phi h / 512 2 \ln 2 \pi^2 r^2 \right] \Sigma \sigma_i \]  \hspace{1cm} (9)

Hence the average power is reduced by the factor 2 \ln 2 which accounts for the fact that the beam shape in the cross radial direction is actually Gaussian, it is not uniform across its width. Therefore the power is not equal at all positions, but rather peaked in the center and rolls off from there in a Gaussian shaped manner.
In this section we will look at the functional form of $\sigma$ for small spheres composed of either water or ice. First consider the interaction of an electromagnetic wave with a spherical drop. The incident radiation induces oscillating electric and magnetic dipoles in the water, associated with the polarization of the dielectric material, in this case, water. The induced dipoles of course constitute the scattered, or re-radiated EM field from the drop. The incident EM wave energy is thus partitioned between the re-radiated field, and the component that is absorbed by the drop. We will return to absorption in a bit.

Dipoles are induced as free charge and dipole moments associated with each molecule are aligned by the incident field. The charge dipoles oscillate at the frequency of the incident radiation. This oscillation of charge produces the field that is scattered from the target.
Mie Theory

A general theory for describing the backscattering of a plane EM wave by a sphere was developed by Mie (1908). From Mie theory,

$$\sigma = \frac{\pi a^2}{\alpha^2} \left| \sum (-1)^n (2n + 1) (a_n - b_n) \right|^2$$

(10)

where $a$ is the drop radius and $\alpha$ is the size parameter, equal to $2\pi a / \lambda$. Here the scattering is $180^\circ$ opposite to the direction of the incident radiation. $a_n$ and $b_n$ are expressed in terms of spherical Bessel and Hankel functions with arguments $\alpha$ and $m$, where $m$ is the complex interaction of refraction,

$$m = n - ik$$

where $n$ is the index of refraction and $k$ is the absorption coefficient.

Physical interpretation of $a_n$ and $b_n$ ??
**Physical Interpretation of** $a_n$ **and** $b_n$

$a_n$ terms represent scattering from dipoles of charge that are induced by the incident field.

Dipoles, quadrapoles, etc. result, as the incident radiation polarizes the dielectric material. At microwave frequencies these charge dipoles are associated with the permanent dipole moment of the water molecule. These dipoles, and quadrapoles can be viewed as the following.

![Diagram](image)

$b_n$ terms represent scattering by induced magnetic dipoles, quadrapoles, etc.
For small values of $\alpha$, the normalized back-cross section increases smoothly with $\alpha$. Note the substantial difference between the ice and water target. This is the dielectric response! Water is a more efficient scatterer than ice due to the fact that charge dipoles are more easily aligned in water than ice. Think of the lattice structure of ice, and strong molecular bonding within.

Note that beyond $\alpha>2$, the backscattering cross section of ice exceeds that of water. This results from the fact that absorption in water exceeds that of ice.

Behavior of $\sigma$ at larger $\alpha$ is highly oscillatory. This oscillatory pattern is associated with Mie scattering, where for example, substantial scattering occurs in the forward direction.
This figure illustrates the relative amount of back- and forward- scatter as a function of the size parameter, $\alpha$. Increasing dominance of the forward scattering lobe is associated with multiple internal reflections of the wave, scattering from higher order dipoles, constructive and destructive interference from these dipoles (constructive interference giving rise to stronger forward scatter, for example), and interference effects of surface waves.

**Figure 5.9** Polar plots of the scattered intensity for selected values of the size parameter. The numbers indicate relative magnitudes in the forward and backward directions. Note the scale change (Bohren and Huffman, 1983).

*From Stephens (1994)*
The Rayleigh Approximation

Consider the form of the back scattering cross section $\sigma$ for small values of $\alpha$, that is, cases where the radius of the particle is small compared to the radar wavelength, $\lambda$. In the limit $\alpha \rightarrow 0$, it is possible to carry out a small argument expansion of the Bessel and Hankel functions that define the $a_n$ and $b_n$ coefficients and express them as polynomials in $\alpha$. In practice, we will neglect all terms of order $\alpha^5$ or higher.

The coefficients are:

$$b_1 \approx -i/45 (m^2 - 1) \alpha^5 + O(\alpha^7)$$

$$a_1 \approx -2i/3 (m^2 - 1/m^2 + 2) \alpha^3 \left[ 1 + 3/5 (m^2 - 2/m^2 + 2) \alpha^2 \right] + O(\alpha^6)$$

$$a_2 \approx -i/15 (m^2 - 1/2m^2 + 3) \alpha^5 + O(\alpha^7)$$

Retaining $\alpha^3$ terms leaves only the first term of $a_1$, the ELECTRIC DIPOLE TERM.

Substituting the $a_1$ expression in (10) yields,

$$\sigma_b = \lambda^2 \alpha^6 / \pi \cdot \left| m^2 - 1/m^2 + 2 \right|^2 (11)$$

This is an expression for the Rayleigh backscattering cross section of a spherical particle, defined for $\alpha \rightarrow 0$. 
The desired transmit spectrum for the CHILL system is indicated by the red curve. The actual (but idealized) transmit spectrum is shown in blue. These “spurs” are undesirable, and are removed by installing a passband filter at the receiver input. In practical applications, this filter has a narrow width. In the example shown, the 3 dB width is 0.75 MHz. The receiver width is narrow to reduce the noise in the receiver.

The shape of the actual transmit pulse is described by a \((\sin x / x)^2\) form.

The noise power of the receiver, which determines the minimum detectable signal of the system, is defined by

\[ N = k T B \]

where \(N\) is power in Watts, \(k\) is Boltzman’s constant, \(T\) is noise temperature set at 290 K by convention and \(B\) is bandwidth in Hertz.
Rayleigh Approximation-con’t

Equation (11) can be rewritten as the following,

\[
\sigma_b = \frac{\pi^2}{\lambda^4} \cdot |K|^2 D^6
\]  

(12)

Note that for a given D, the Rayleigh back scattering cross section is proportional to \(1/\lambda^4\). Why should this behavior be expected?

Index of refraction

The complex index of refraction m is a function of both \(\lambda\) and \(T\), for any given substance. Table 4.1 from Battan (1973) provides requisite values.

| Table 4.1 | The Components of the Complex Index of Refraction, \(|K|^2\), and the Imaginary Part of \((-K)\) of Water as Functions of Temperature and Wavelength |

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Temperature (°C)</th>
<th>10</th>
<th>3.21</th>
<th>1.24</th>
<th>0.62</th>
</tr>
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<tbody>
<tr>
<td>(\kappa)</td>
<td>20</td>
<td>8.88</td>
<td>8.14</td>
<td>6.15</td>
<td>4.44</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>9.02</td>
<td>7.80</td>
<td>5.45</td>
<td>3.94</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>8.99</td>
<td>7.14</td>
<td>4.75</td>
<td>3.45</td>
</tr>
<tr>
<td></td>
<td>-8</td>
<td></td>
<td>6.48</td>
<td>4.15</td>
<td>3.10</td>
</tr>
<tr>
<td>(\kappa)</td>
<td>20</td>
<td>0.83</td>
<td>2.00</td>
<td>2.86</td>
<td>2.59</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.90</td>
<td>2.44</td>
<td>2.90</td>
<td>2.37</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1.47</td>
<td>2.89</td>
<td>2.77</td>
<td>2.04</td>
</tr>
<tr>
<td></td>
<td>-8</td>
<td></td>
<td></td>
<td>2.55</td>
<td>1.77</td>
</tr>
<tr>
<td>(</td>
<td>K</td>
<td>^2)</td>
<td>20</td>
<td>0.928</td>
<td>0.9275</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.9313</td>
<td>0.9282</td>
<td>0.9152</td>
<td>0.8726</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.9340</td>
<td>0.9300</td>
<td>0.9055</td>
<td>0.8312</td>
</tr>
<tr>
<td></td>
<td>-8</td>
<td></td>
<td></td>
<td>0.8902</td>
<td>0.7921</td>
</tr>
<tr>
<td>Im((-K))</td>
<td>20</td>
<td>0.00474</td>
<td>0.01883</td>
<td>0.0471</td>
<td>0.0915</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.00688</td>
<td>0.0247</td>
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<td>0.1142</td>
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<tr>
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<td>0.01102</td>
<td>0.0335</td>
<td>0.0807</td>
<td>0.1441</td>
</tr>
<tr>
<td></td>
<td>-8</td>
<td></td>
<td></td>
<td>0.1036</td>
<td>0.1713</td>
</tr>
</tbody>
</table>

Source: Gunn and East 1954.
Calculation of $|K|^2$

We need to take the modulus of $K$, and then square the result to find the numerical value of $|K|^2$. Hence,

$$|K|^2 = |n^2 - 2\text{i}nk - k^2 - 1 / n^2 - 2\text{i}nk - k^2 + 2|^2$$

To proceed, multiply top and bottom by the complex conjugate of the denominator, $n^2 + 2\text{i}nk - k^2 + 2$.

Hence

$$|K|^2 = [(n^4 + 2nk^2 + n^2 + k^4 - k^2 - 2)^2 + 36n^2k^2] / [(n^2 - k^2 + 2)^2 + 4n^2k^2]^2$$

Since this is a real number, values of $n$, $k$ when inserted, yield the desired value for the dielectric factor, $K$. For most purposes, $|K|^2 = 0.93$.

For a spherical particle, $M = 4/3 \pi a^3 \rho$, where $\rho$ is the particle density, (12) can be written as,

$$\sigma_b = 36 \pi^3 / (\lambda^4 \rho^2) \cdot |K|^2 M^2 \quad (13)$$

which again is only valid for Rayleigh conditions. In a moment we will examine the relationship between particle size and wavelength such that the Rayleigh approximation is a valid assumption.
Dielectric Factor for Ice

Ice is more complicated since the density of ice is variable. Ice density may range from 0.92 g/cm³ for pure ice, to 0.05 g/m³ for loosely bound aggregates. From Table 4.2, for $n = 1.78$ and density equal to 1.0 g/cm³, $K^2 = 0.197$. What is the value of $K^2$ for smaller ice densities? Note that the absorption coefficient in ice is much smaller than that for water. Consequently, attenuation by ice particles, especially snow and cloud ice particles, is negligibly small at most meteorological radar wavelengths.

| TABLE 4.2 | The Components of the Complex Index of Refraction, $|K|^2$, and the Imaginary Part of $-K$ of Ice as Functions of Temperature |
|-----------------|------------------------------------------------------------------------------------------------------------------|
| **Quantity**    | **Temperature (°C)** | **Value** |
| $n$ ...          | All temperatures when $\rho = 0.92 \text{ gm/cm}^3$ | 1.78 |
| $\kappa$ ....    | $|0\ 
\ 
-10\ 
\ 
-20|$ | $2.4 \times 10^{-3}$ $7.9 \times 10^{-4}$ $5.5 \times 10^{-4}$ |
| $|K|^2$ ...       | All temperatures when $\rho = 1$ | 0.197 |
| $\text{Im}(-K)$ | $|0\ 
\ 
-10\ 
\ 
-20|$ | $9.6 \times 10^{-4}$ $3.2 \times 10^{-4}$ $2.2 \times 10^{-4}$ |

Source: Gunn and East 1954.

Battan (1973)
Dielectric for Ice-con’t

If we return to equation (13) of the form,

\[ \sigma_b = 36 \pi^3 / (\lambda^4 \rho^2) \cdot |K|^2 M^2 \quad (13) \]

we can use the Debye formula,

\[ (K/\rho) M_T = (K_i/\rho_i) M_i + (K_a/\rho_a) M_a \quad (14) \]

for an ice, water mixture, such as an aggregate. Here K represents the respective dielectric constants for the ice and air components. \( M_T \) represents the total mass of the particle. The second term may be neglected since \( K_a \) is small, as is the fraction of air in the mixture, \( M_a \). Hence (14) may be approximated as,

\[ (K/\rho) M_T \approx (K_i/\rho_i) M_i \]

Since \( M_T \) is approximately equal to \( M_i \), we see that \( K/\rho \) is nearly constant for ice-air mixtures. As the density of ice varies between 0.22 and 0.92 g/cm\(^3\), \( K/\rho \) varies between 0.5 and 0.46. This implies we may use a fixed value of \( K^2 \) for ice, 0.197. We may also use the dielectric factor for water in equation (12), and set \( D \) to be the equivalent melted diameter of the particle as a result of complete melting of the ice particle.
Rayleigh-Mie Transition

Let us consider the transition from Rayleigh to Mie scattering, or the relationship between particle size and wavelength at which the Rayleigh approximation is no longer valid. Gunn and East (1954) examined the ratio of the Mie scattering cross section to the Rayleigh backscattering cross section at several radar wavelengths, as a function of \( \alpha = \frac{2\pi a}{\lambda} \).

Here \( \delta \) is the ratio of the Mie scattering cross section to the Rayleigh backscattering cross section. \( \delta = \frac{\sigma_{\text{mie}}}{\sigma_{\text{rayleigh}}} \)

Lets define the Rayleigh approximation valid for \( \alpha < 0.22 \) (see figure 4.3 to the left).

There are some differences in this definition in the literature, for example, some authors define the Rayleigh region for \( 1/2 \leq \delta \leq 2 \).

Battan (1973)
Rayleigh region

For the Rayleigh approximation valid for $\alpha < 0.22$, using $\alpha = \frac{2\pi a}{\lambda}$, we have $D < 0.07 \lambda$ for the Rayleigh region.

For S band (10 cm wavelength) $D < 7$ mm
For C band (5 cm wavelength) $D < 3.5$ mm
For X band (3 cm wavelength) $D < 2.1$ mm
For K band (1 cm wavelength) $D < 0.7$ mm
For W band (0.4 cm) $D < 0.25$ mm

For S band, all particles but large hailstones are in the Rayleigh region. However, for shorter wavelengths, small raindrops are in the Mie region, as well as many ice particles. This is certainly true at cloud radar wavelengths (K and W band).

Departure from Rayleigh conditions may be attributed to both scattering and absorption processes. That is, the backscattering cross section may be reduced due to increased forward scatter, or increased absorption by the particle. Size and the type of dielectric involved here are both important factors. For ice, the departure from Rayleigh backscatter is due primarily to scatter (since ice is a weak absorber). For water, absorption is substantial and also contributes to departures from Rayleigh conditions.
Reflectivity Factor

We can now summarize our relationships between average return power and reflectivity $\eta$.

$$P_r = [P_t G^2 \lambda \theta \phi h / 512 2\ln 2 \pi^2 r^2] \eta \quad (15)$$

When Rayleigh conditions apply, the sum of the backscattering cross sections per unit volume can be replaced by,

$$\eta = \Sigma \sigma_i = \pi^5 \lambda^{-4} |K|^{2 \Sigma D_i} \quad (16)$$

resulting in the following relationship for Rayleigh conditions,

$$P_r = [\pi^3 P_t G^2 \theta \phi h / 512 2\ln 2 \lambda^2] (1/r^2) |K|^{2 \Sigma D_i} \quad (17)$$

This expression is ideal in the sense that it applies only to radar systems that have no power losses of any sort, including atmospheric attenuation (by both gases and particles), and power losses between the received signal power (say at the antenna) and the “front end” of the receiver, and losses within the receiver itself. We will quantify each of these losses as we go on.

Let us now offer an important definition, for the radar reflectivity factor $Z$ (for Rayleigh conditions).

$$Z = \Sigma D_i \quad (18)$$
Reflectivity Factor, con’t

The summation over $D_i$ results in $Z$ (for Rayleigh conditions), whereas the summation over $\sigma$ yields the reflectivity, $\eta$. Note that $Z$ is therefore independent of wavelength. In terms of a particle size distribution, we have

$$Z = \sum n_i D_i^6$$  \hspace{1cm} (19)

where $n_i$ is the particle concentration in the $i^{th}$ size category. $Z$ has specific units of $\text{mm}^6 / \text{m}^3$. Hence $D$ is expressed in mm, and the particle concentration in $\text{m}^{-3}$. Using $Z$, we can rewrite

$$P_r = \left[ \pi^3 \rho_\text{t} G^2 0\phi c \tau / 1024 \ln 2 \lambda^2 \right] (1/r^2) \left| K \right|^2 Z$$ \hspace{1cm} (20)

In (20), $c$ represents the speed of an EM wave in a vacuum, and $\tau$ is the pulse length. Eqn. (20) can be written in a more compact form as,

$$P_r = C (1/r^2) \left| K \right|^2 Z$$ \hspace{1cm} (21)

where $C$ is defined as the radar constant, since it contains only constants and engineering terms related to the radar. Again, this is for a “no loss” radar system. When Rayleigh conditions do not apply,

$$P_r = C (1/r^2) \left| K \right|^2 Z_e$$ \hspace{1cm} (22)

where $Z_e$ is the equivalent reflectivity factor. In practice, $K$ is that for water targets so we are really always dealing with $Z_e$. 
If we take $10 \log_{10}$ of both sides of Eqn. (22), we have,

$$10 \log_{10} P_r = 10 \log_{10} C + 10 \log_{10} (0.93) - 20 \log_{10} r + 10 \log_{10} Z_e \quad (23)$$

measuring $P_r$, and range to target $r$, we can then solve (23) for $Z_e$. Furthermore, we define,

$$\text{dB}Z_e = 10 \log_{10} Z_e \quad (24)$$
Accounting for Power Losses

Real radar systems have power losses due to coupler losses, waveguide losses and receiver losses. The latter arise due to the fact that the receiver has a finite bandwidth (typically 20-30 MHz) to limit the noise power of the receiver. This is done by using a filter that passes just this passband. The characteristics of this filter are not exactly matched to that of the transmitter. Hence the transmit pulse may be slightly broader than the receiver passband, resulting in a slight power loss. Defining $L_m$ as the power loss associated with antenna, coupler and waveguide losses, and $L_r$ the loss in the receiver, eqn. (20) can be written as,

$$ P_r = \left[ \pi^3 P_t G^2 \theta \phi c \tau L_m L_r / 1024 \ln 2 \lambda^2 \right] (1/r^2) \left| K \right|^2 Z $$  

(25)

In (25), $L_m$ and $L_r$ as expressed as dimensionless numbers less than unity. Express all units in the MKS system to use (25). Radians are used for the antenna beamwidth inputs.
Ice-Water Differences—Why?

The returned power from water targets can be written as  \[ P_w = C K_w^2 Z / r^2 \]
The returned power from ice targets can be written as  \[ P_i = C K_i^2 Z / r^2 \]
Since the dielectric factors for water and ice are 0.93 and 0.21 respectively, why is  \( P_w > P_i \)?

Consider a vertically polarized EM wave incident on a spherical target. Incident wave polarizes the dielectric by inducing many small charge dipoles in the droplet (or ice particle). The sum of these induced dipoles constitutes the polarization vector, \( \mathbf{P} \) in the dielectric. \( \mathbf{P} \) lies parallel to the incident \( \mathbf{E} \) field vector. These induced dipoles oscillate at the frequency of the incident wave, and thus radiate, constituting the backscattered field for the case of Rayleigh targets.

The strength of the induced dipole (in a macroscopic sense), is given by the product of the incident field and the “polarizability” of the dielectric. The polarizability of water is larger than ice since the water molecules can move more freely than ice molecules. **Polarization** of some material is the property that relates to the ability of the material to form dipoles when excited by an incident \( \mathbf{E} \) field. Of course this process is amplified when the dielectric possesses a permanent dipole moment, as is the case for the water molecule. **Polarization is defined as the ability of the local charge distribution associated with atoms and molecules in the material to align to an incident field.**
Polarization

Stephens (1994)

The polarization can be expressed as \( \mathbf{P} = (\varepsilon_r - 1) \varepsilon_0 \mathbf{E} \) where \( \varepsilon_r \) is the relative permittivity, and \( \varepsilon_0 \) is the permittivity of a vacuum. Hence the permittivity is like a constant of proportionality that relates the polarization of a dielectric to the strength of the incident field. The relative permittivity is a complex number of the form

\[
\varepsilon_r = \varepsilon_r' + i \varepsilon_r''
\]

with \( \varepsilon_r' = n^2 - k^2 \) and \( \varepsilon_r'' = 2nk \)

For weakly absorbing media, \( n = \sqrt{\varepsilon_r'} \)

Let’s take a look at this polarization effect as a function of frequency.
Polarization-Frequency Dependence

Consider EM waves at various frequencies incident on water droplets. Consider higher frequencies (say ultraviolet), moderate frequencies (say infrared), and lower frequencies (say microwave).

**Ultraviolet**

In this case the lightest portion of the matter can only respond, electrons. Hence electronic transitions results (orbital changes). Only the lightest components of the materials can “respond” to the highest frequencies.

**Infrared**

In this case, denser portions of the matter can respond, atoms. Therefore oscillations are more sluggish and frequencies associated with atomic transitions are lower than those associated with electronic transitions.

**Microwave**

Slowest oscillations occur when the molecular dipole moments respond to the incident E field. In this case, the most massive components of the dielectric (molecules) respond to E fields with the lowest frequencies (microwave).
Temperature Affects on Dielectric Response

Because water drops and ice particles are at finite temperature, thermal agitation provides for molecular motions against the effects of viscosity. The Debye relaxation for a sphere of radius a is defined as

\[ \tau = 4\pi \eta a^3 / k T \]

where \( \eta \) is the viscosity of the medium, k is Boltzman’s constant and T is temperature. When \( \tau \) is large, (for media with large viscosities) the response of the dielectric to incident EM field is sluggish; when \( \tau \) is small (for low viscosity media), the response is large (since thermal agitation provides enough molecular kinetic energy to buffer the effects of viscosity). Since \( \eta \) is smaller in water compared to ice, the Debye relaxation for water is less than that for ice. Correspondingly molecular dipole moments in water respond to incident fields, that is, the dipole moments in water are highly polarized by the incident field. Hence \( \varepsilon_r' \) is large, which implies that the real part of the index of refraction, n, is large. For ice, there is less polarization, \( \varepsilon_r' \) is less and hence “n” is less.

What would be the low temperature limit of the Debye relaxation?

\( \tau \) is the “response time” of a dielectric to an incident E field. This concept applies to dense media only.

Debye relaxation does not apply to a gas since molecular/atomic interactions are negligible. See Owens, J.C., (1967), Appl. Opt., 6, pp. 51-59. (n = p/T)
<table>
<thead>
<tr>
<th>Type of interaction</th>
<th>n (real component)</th>
<th>k (imaginary component)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Non-resonant or Scattering</td>
<td>Resonant or Absorption</td>
</tr>
<tr>
<td></td>
<td>Dipoles align to E field</td>
<td>Natural vibrational or rotational frequencies</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Wavelength or Frequency dependence</th>
<th>Decreases monotonically with decreasing wavelength</th>
<th>Max in k occurs at resonant frequency $\omega = \omega_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Dipoles have time to respond to slowly oscillating field; larger $n$ at longer wavelength</td>
<td>Resonant freq characteristic of all matter</td>
</tr>
<tr>
<td></td>
<td>Smaller $n$ at shorter wavelength (dipoles respond less to high freq)</td>
<td>Absorption peak is broad due to interaction between dipoles</td>
</tr>
</tbody>
</table>

| T dependence | For fixed wavelength, $n$ increases for increasing $T$ (thermal agitation allows dipoles to react more readily) | “Broadening” of $k$ increases with decreasing $T$ (viscosity increases) |
Application of Rayleigh Scatter-Rainfall Estimation

One of the first applications of reflectivity $Z_e$ is to use this measurement to estimate rain rates. Obviously since this is a power based measurement, any process that adds uncertainty or error to the estimation of reflectivity will of course affect the rain rate estimate. Obviously calibration errors are important, as is attenuation, the presence of ice particles such as hail in the pulse volume, artificial increases in reflectivity due to melting processes (radar bright band), partial beam filling at longer ranges, increase in beam height above ground with slant range, and lastly, variability in the drop size distributions from those that are assumed in the development of power law relationships between reflectivity and rain rate.

Let us assume for this discussion that the size of raindrops follows an inverse exponential size distribution of the form,

$$N(D) = N_o e^{-\lambda D} \quad (26)$$

where $N_o$ is the slope intercept ($m^{-4}$), $D$ is diameter and $\lambda$ the slope of the distribution ($m^{-1}$). For a wide range of range rates, $N_o = 8 \times 10^{-6}$, $m^{-4}$. Knowing the size distribution and the definition of $Z$, we can derive an expression for $Z$ as follows:

$$Z = \int_0^\infty N(D) D^6 \, dD = \int_0^\infty N_o e^{-\lambda D} D^6 \, dD \quad (27)$$
Regime Variability

Theon (1992)

FIGURE 12. Reflectivity-rain relationships vary substantially for differing rain systems at the lower rain rates (personal communication, 1991, D. Rosenfeld, Hebrew University, Jerusalem, Israel).