Error Estimation in Wind Fields Derived from Dual-Doppler Radar Measurement

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ABSTRACT

Variance in horizontal and vertical winds is predicted when these components are computed from dual-Doppler velocity measurements combined with terminal velocity estimates and the continuity equation. Errors in horizontal wind magnitude and direction are shown to be functions of wind direction and speed as well as spatial location. Vertical wind could be estimated with errors less than a few meters per second up to altitudes near 14 km over a region 4dX4d, where 2d is the radar separation. Vertical wind variance at high altitudes is related to accumulation of errors due to the integration of the continuity equation. The cause of wind variance is assumed to be uncertainty in mean Doppler velocity estimates produced by spectrum broadening mechanisms (e.g., shear, turbulence). Two interpolation methods, used to estimate Doppler velocity at common grid locations, are compared and their contribution to Doppler velocity variance reduction is calculated. Terminal velocity variance has been related to uncertainties in droplet-size distributions and reflectivity estimate variance. The methods derived herein are applied to determine the errors in wind speeds calculated from dual-Doppler data.

1. Introduction

Wind fields can be inferred by combining Doppler velocities, measured by two spaced radars, with particle terminal velocity estimates and subsequently solving the continuity equation (Lhermitte, 1970; Miller and Strauch, 1974; Ray et al., 1975). Wind field determination is greatly simplified if synthesis is performed in cylindrical coordinates with an axis colinear with the line connecting the two radars. Lhermitte and Miller (1970) suggested that Doppler data acquisition be confined to planes in this frame (COPLAN method) so that winds in each plane could be deduced directly with minimal data interpolation. Irrespective of the acquisition mode, wind synthesis is facilitated when executed in cylindrical coordinates and Cartesian wind components are derived from the synthesized cylindrical components. We shall determine wind estimate variance assuming this synthesis procedure. Although we can solve directly for Cartesian wind components, this necessitates a solution of a linear, inhomogeneous, hyperbolic partial differential equation to derive vertical wind (Armiio, 1969). Lhermitte and Miller (1970) estimated errors in dual-Doppler derived horizontal wind for regions where the particle’s vertical motion can be neglected. In this paper, error in both horizontal and vertical wind related to interpolated mean Doppler velocity variance and error reduction associated with interpolation is evaluated. Further, vertical velocity variance is related to height-dependent mass density and station separation.

To determine wind we must interpolate Doppler data to a grid common to both radars. We analyze two interpolation methods: 1) linear 4-point interpolation appropriate for COPLAN data acquisition and 2) distance-weighted spatial averaging applicable to data acquired using azimuthal sector scans stepped in elevation angle.

2. Wind estimate variance (cylindrical components)

Although we seek Cartesian component velocity variance for dual-Doppler derived wind, we first evaluate the variance of the wind components illustrated in the cylindrical coordinate system depicted in Fig. 1. Following Miller and Strauch (1974) the particle velocity at a grid point in a plane elevated by angle $\alpha$ is specified by

$$U_x(\rho, \phi) = \frac{r_1(s+d)V_1 - r_2(s-d)V_2}{2dp},$$

$$U_z(\rho, \phi) = \frac{r_2V_2 - r_1V_1}{2d},$$

where $V_1, V_2$ are the interpolated estimates of mean
(i.e., pulse-volume averaged) Doppler velocities measured by radars 1, 2, and $U_p$, $U_s$ are particle velocity components perpendicular and parallel to the baseline. The wind field is then

$$W_p = U_p - V_1 \sin \alpha, \quad W_s = U_s,$$  \hspace{1cm} (2.2)

where $V_1$ is the particle's interpolated mean terminal velocity estimate. Because $V_1$, $V_2$ are composed of independent measurements, $W_p$ and $W_s$ variances are directly determined from

$$\text{var}[W_p] = \sigma_z^2 = \frac{r_1^2 (s + d)^2 \sigma_1^2 + r_2^2 (s - d)^2 \sigma_s^2}{4d^2 \rho^2},$$

$$+ \sigma_z^2 \sin^2 \alpha,$$  \hspace{1cm} (2.3a)

$$\text{var}[W_s] = \sigma_z^2 = \frac{r_s^2 \sigma_z^2 + r_1^2 \sigma_1^2}{4d^2},$$  \hspace{1cm} (2.3b)

where $\sigma_z^2$, $\sigma_1^2$, $\sigma_s^2$ are variances of $V_1$, $V_2$, $V_s$. Wind perpendicular to the plane is determined by solving the continuity equation

$$\nabla \cdot \gamma \hat{W} = 0,$$  \hspace{1cm} (2.4)

where $\gamma = \gamma_0 \exp(-I \rho \sin \alpha)$ is the mass density. Solving (2.4) we have

$$W_a = \frac{1}{\gamma} \int_0^a \left[ \frac{\partial}{\partial \rho} (\rho W_\rho) + \rho \frac{\partial}{\partial s} (W_s) \right] d\alpha,$$  \hspace{1cm} (2.5)

in which $W_a(\rho, 0, s) = 0$ has been used as a boundary condition. Now the variance in $W_a$ is

$$\text{var}[W_a] = \sigma_z^2 = \frac{1}{\gamma^2} \left[ \text{var} \left[ \int_0^a \frac{\partial}{\partial \rho} (\rho W_\rho) d\alpha \right] + \text{var} \left[ \int_0^a \rho \frac{\partial}{\partial s} (W_s) d\alpha \right] \right],$$  \hspace{1cm} (2.6)

where the terms in (2.5) are assumed uncorrelated. This assumption is satisfied if contiguous interpolated grid values $V_1$ (or $V_2$) are uncorrelated and if the finite-difference approximation used to evaluate the partial derivatives does not use common grid values. Evaluating the first term in (2.6) using finite differences we obtain the approximate expression

$$\frac{1}{\gamma^2} \text{var} \left[ \int_0^a \frac{\partial}{\partial \rho} (\rho W_\rho) d\alpha \right] \approx \frac{1}{(2\gamma \Delta \rho)^2} \int_0^a \gamma (W_{\rho 2} - W_{\rho 1}) d\alpha,$$  \hspace{1cm} (2.7)

where $2\Delta \rho = \rho_2 - \rho_1$ is twice the grid spacing $\Delta \rho$, and $W_{\rho 2}$, $W_{\rho 1}$ are radial wind components evaluated at $r = \rho_1$ and $r = \rho_2$.

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1 Throughout this paper mean values will always refer to pulse-volume averages.

\[\begin{figure}
\centering
\includegraphics[width=\textwidth]{dual-doppler-coplan-fig1}
\caption{Dual-Doppler COPLAN coordinate system.}
\end{figure}\]

\[\rho_1, \rho_2. \text{ We assumed } \rho > \Delta \rho \text{ and that } \gamma(\rho_2) = \gamma(\rho_1). \text{ Considering the integral to be a sum of terms having independent errors, (2.7) reduces to}
\]

$$\frac{2\rho \Delta \alpha}{(2\gamma \Delta \rho)^2} \int_0^a \gamma^2 \sigma_z^2 d\alpha,$$  \hspace{1cm} (2.8)

where $\Delta \alpha$, the step size used in the integration of $\Delta W_\rho$, is the spacing between grid planes. Even though $\sigma_z^2$ is a function of $\rho$, $\alpha$ and $s$, we assumed $\sigma_z^2$ equal to $\sigma_1^2$, a reasonable approximation when variance changes across $\Delta \rho$ are small. Applying similar methods to evaluate the second term in (2.6) and substituting this and (2.8) into (2.6), we have $\sigma_z^2$ given by

$$\sigma_z^2 = \frac{2\Delta \rho^2}{(2\gamma \Delta \rho)^2} \left[ \int_0^a \gamma^2 \sigma_z^2 d\alpha + \int_0^a \gamma^2 \sigma_z^2 d\alpha \right],$$  \hspace{1cm} (2.9)

in which we equated $\Delta \rho$ and $\Delta \alpha$. Substituting (2.3) into (2.9), we obtain the solution

$$\sigma_z^2 = 2\alpha \Delta \rho \gamma^2 \sigma_1^2 \Gamma \frac{r_1 r_2}{4(\Delta \rho)^2} C_1 + C_2,$$  \hspace{1cm} (2.10)

for variance in the $z$ component of wind, where

$$C_1 = \frac{1}{\gamma^2 \Delta \rho} \int_0^a \gamma^2 (\sigma_z^2 + \sigma_s^2) d\alpha, \quad C_2 = \frac{1}{\gamma^2 \Delta \rho} \int_0^a \gamma^2 \sigma_s^2 \sin^2 \alpha d\alpha.$$

The mass-density weighted variance factors $C_1$ and $C_2$ have values that depend upon height. Assuming $\alpha$ small and $\sigma_z$, $\sigma_s$ to be $\alpha$-independent, $C_1$ is given by the approximate formula

$$C_1(\rho \alpha) \approx (\sigma_z^2 + \sigma_s^2) F_1(\rho \alpha),$$

where $F_1(\rho \alpha) = 1 - \frac{\exp\left(2\Gamma \rho \alpha\right) - 1}{2\Gamma \rho \alpha}$. Thus for a constant arc height $\rho \alpha$, $F_1$ is a constant. Eq. (2.11) shows that $C_1$ increases monotonically from $\sigma_z^2 + \sigma_s^2$ at $\rho \alpha = 0$ to nearly four times this value for an arc height of 10 km (for $\Gamma$ = 0.113 km$^{-1}$). Therefore decreasing mass density contributes significantly to $\sigma_z^2$. In like manner we can obtain an approximate expression for $C_2$. However, for fixed arc height, $C_2$
is proportional to \( \alpha^2 \) and hence is much less than unity for \( \alpha \) small. Eqs. (2.10) and (2.11) show that variance increases faster than linearly with \( \alpha \) because \( C_4 \) and \( C_2 \) both increase with \( \alpha \).

We do not make any numerical assessment of cylindrical wind component variance but instead use the above formulas to infer Cartesian wind component variance (Section 4).

3. Interpolated Doppler velocity variance

The previously derived formulas provide estimates of synthesized wind variance, given \( \sigma_1^2 \), \( \sigma_2^2 \). Herein we shall relate \( \sigma_1^2 \) to mean Doppler estimate variance, \( \sigma^2(\bar{V}) \), grid point range, wind shear and antenna beam width. Mean Doppler estimate variance at a grid point is decreased through interpolation by an amount proportional to a variance reduction factor \( R^2 \). That is,

\[
\sigma_1^2 = (1-R^2)\sigma^2(\bar{V}),
\]

where \( \sigma^2(\bar{V}) \) is assumed constant over the interpolation domain. \([In Appendix A, (1-R) is derived for different interpolation schemes but is, approximately, inversely proportional to the number of mean Doppler estimates interpolated.] The \( \sigma^2(\bar{V}) \) is a function of Doppler spectrum width, signal-to-noise ratio (S/N), beam pointing precision, etc. Spectrum width \( \omega_s \) depends on beamwidth, wind shear, turbulence, and many other factors (Nathanson, 1969 p. 206). Turbulence and wind shear are the dominant spectrum-width-producing mechanisms (Sirmans and Doviak, 1973), although at range more than 30 km shear is expected to predominate (Nathanson, 1969). However, Waldteufel (1976) shows that within a severe storm the shear contribution is small compared to other broadening factors. We now assess errors in \( V \) and gauge the magnitudes of \( \sigma^2(\bar{V}) \) and \( \sigma^2(\bar{V}_i) \), the variance in mean terminal velocity estimates.

a. Systematic errors

Both systematic and random errors contribute to \( \bar{V} \). Although systematic (bias) errors are introduced by certain Doppler processing techniques (Sirmans and Bumgarner 1975), we assume these errors can be ignored. However, causes for other bias errors include 1) non-uniform reflectivity \( Z \) within a radar sample volume, 2) use of incorrect \( V_0 \), \( Z \) relationship (Section 3c), 3) inaccuracies in beam position. For the latter cause, Doppler bias error \( \delta_s \), due to an antenna azimuthal (or elevation) position inaccuracy \( \delta \phi \) (rad), for small elevation is

\[
\delta_s = K_s \delta \phi,
\]

where \( K_s \) is the azimuthal (or vertical) wind shear of the Doppler velocity. Assuming \( 5 \times 10^{-4} \) s\(^{-1}\) for \( K_s \)

\[^1\]The caret is used to designate single data sample estimates of, for example, mean Doppler velocity.

(Donaldson et al., 1972) a 0.2° error causes a Doppler bias error of about 1 m s\(^{-1}\) for a \( 60 \) km distance. Using (2.1 a, b) we note that in this case bias error \( \delta \), produces spatially dependent systematic errors in \( W_s \), \( W_a \), but may not contribute significantly to error in \( W_a \) because bias errors tend to cancel in the partial derivative of (2.5).

b. Random errors

Doppler spectrum width relates to random errors in mean Doppler estimates. For example, assuming radial velocity shear as the dominant width-producing mechanism. Sirmans and Doviak (1973) showed, for small elevation angles and Gaussian-shaped antenna pattern, that spectrum width is given by

\[
\omega_s = 0.3 K_s \delta \phi \angle 
\]

where \( K_s \) is the transverse shear of radial velocity, \( \angle \) the pulse range, and \( \delta \phi \) the one-way 3 dB beamwidth (rad). The interpolated mean Doppler velocity variance is defined as

\[
\sigma_1^2 = (1-R^2)\frac{\omega_s^2}{N_i},
\]

where \( N_i \), the equivalent number of independent echo samples coherently processed, is given by

\[
N_i = \frac{w_s^2}{\sigma^2(\bar{V})}.
\]

Because \( \bar{V} \)'s derived from the argument of signal covariance for time lag \( T \) (\( T \) is the pulse repetition time) have minimum variance for covariance estimated from a sequence of uncorrelated sample pairs (Miller and Rochwarger, 1972), we restrict our attention to velocities estimated by that technique. Berger and Groginsky (1973) demonstrate that, when covariance is an average of estimates from contiguous sample pairs comprising a uniform train of samples as is usually obtained from pulse Doppler radars, variance at large S/N ratio is simply

\[
\sigma^2(\bar{V}) = \frac{w_s V_m}{2 \pi M} \quad \text{for } w_s \leq \frac{V_m}{\pi},
\]

where \( V_m \) is the maximum unambiguous Doppler velocity (i.e., \( V_m = \lambda/4T \)) and a Gaussian-shaped spectrum is assumed. Thus the interpolated velocity variance is

\[
\sigma_1^2 = \frac{0.3 K_s \delta \phi \Delta V}{4 \sqrt{\pi}} (1-R^2),
\]

where \( \Delta V = 2V_m/M \) is the Doppler velocity resolution. To estimate \( \sigma_1 \), we assume \( K_s = 10^{-4} \) s\(^{-1}\), \( \angle = 60 \) km and \( \delta \phi = 1^\circ \). We obtain from (3.3) a value \( w_s = 3 \) m s\(^{-1}\).
and if $\Delta V = 1 \text{ m s}^{-1}$, then $\sigma(\bar{V}) = 0.63 \text{ m s}^{-1}$. Now if $R^2 = 0.9$ (i.e., assuming that about 10 independent $\bar{V}$ estimates are averaged) then $\sigma_1$ is $0.2 \text{ m s}^{-1}$.

Random errors in the beam positioning angle $\delta \phi$ will generate additional random errors in Doppler velocity as is evident from (3.2). For example, an rms error of 0.1° will result in a 1 m s$^{-1}$ rms error in $\bar{V}$ for the conditions assumed above. This value compares with the rms error in $\bar{V}$ due to $w_r$. Random errors in $\bar{V}$ can be reduced by increasing the number of echo samples coherently processed, but random errors due to other sources (e.g., $\delta \phi$) are not decreased by this processing and may require variance reduction by interpolation. Because a $10^{-2}$ s$^{-1}$ wind shear can be maintained over large regions of severe storms, errors in beam pointing may place an upper limit on radar dwell time (i.e., the number of radar pulses processed coherently). For example, a 10 cm Doppler radar with $V_m = 32$ m s$^{-1}$ needs to process no more than 64 samples to achieve a Doppler estimate variance less than that produced by a 0.1° rms beam pointing error.

c. Terminal velocity variance

To determine wind, a $V_t$ estimate must be made. An estimate can be derived from application of a $V_t, Z$ relationship, adjusted for air density variation with height. The relationship (Atlas et al., 1973)

$$V_t = 2.65 Z^{0.11} \left( \frac{\rho}{\rho_0} \right)^{0.4} \text{ [m s}^{-1}]$$

(3.8)

well represents experimental data over a large range of $Z$ (i.e., $1 \leq Z \leq 10^5 \text{ mm}^3 \text{ m}^{-3}$) for regions of liquid water. Terminal velocity variance $\sigma^2(\bar{V}_t)$ is caused by reflectivity estimate variance $\sigma^2(\bar{Z})$ and variance in the $V_t, Z$ relation. We now compare errors from these two mechanisms.

Since Eq. (3.8) shows $V_t$ is a slowly varying function of $Z$, we can relate $\sigma^2(\bar{V}_t)$ to $\sigma^2(\bar{Z})$ (Papoulis, 1965, p. 212) in the form

$$\sigma^2(\bar{V}_t) = \left[ \frac{d V_t}{d \bar{Z}} \sigma(\bar{Z}) \right]^2 \text{ [m s}^{-2}]$$

(3.9)

where the derivative is evaluated at the mean $\bar{Z}$. Sirmans and Doviak (1973) have shown for logarithmic receivers that $\sigma^2(\bar{Z})$ may be written

$$\sigma^2(\bar{Z}) = \frac{1.28 Z}{K},$$

(3.10)

where $K$, the equivalent number of independent intensity estimates, is given approximately by

$$K = \frac{\sqrt{\pi \rho_0 M}}{V_m}.$$  

(3.11)

Thus

$$\sigma^2(\bar{V}_t) = \frac{0.116 (\bar{Z})^{-0.772}}{K}.$$  

(3.12)

Note that the number of independent velocity estimates $N_i$ as computed from (3.5) is two times larger than $K$; the difference is related to the minimum variance properties of the velocity estimates. Using previously specified values for the parameters and assuming $\bar{Z} > 1$ we find that $\sigma(\bar{V}_t)$ is less than 0.13 m s$^{-1}$.

Eq. (3.11) gives $\sigma^2(\bar{V}_t)$ associated with reflectivity estimate uncertainty. However, the results of Joss and Waldvogel (1970) show variability in drop-size distributions within a storm produce standard deviations in terminal velocity nearer 1 m s$^{-1}$ independent of $Z$. Therefore unless an accurate measure of drop-size distribution can be made we shall assume $\sigma(\bar{V}_t)$ equal to 1 m s$^{-1}$. Moreover, if the drop-size distribution is invariant of data location within the interpolation domain (i.e., over 1–2 km), the errors in $\bar{V}_t$ will not benefit reduction from interpolation (Appendix A). Variance reduction results only if $\bar{V}_t$ estimates are uncorrelated, a condition not expected with errors due to uncertainties in the $V_t, Z$ relation. Therefore we assume $\sigma = \sigma(\bar{V}_t) = 1$ m s$^{-1}$.

Larger errors, up to several meters per second, in $V_t$ estimates can be caused by erroneously relating regions of hail with a $V_t, Z$ relation appropriate for liquid water. Usually there is little or no information to uniquely identify these regions, and bias errors $\delta V_t$ will result in an error in $W_a$ although not in $W_s$ [Eq. (2.2)]. If $\delta V_t$ is $\rho$-independent, the bias error in $W_a$ tends to cancel because of differentiation in (2.5). We will now show, for typical dual-Doppler geometry (i.e., $\alpha \lessgtr 1$ rad), that the error in $W_a$ is significantly smaller than $\delta V_t$.

Assume $V_t$ has the spatial dependence of (3.8) and substitute $V_t + \delta V_t$ for $V_t$ in (2.2) and the result into (2.5). It can be shown that the bias error $\delta W_a$ for small $\alpha$ is

$$\delta W_a = -\frac{\delta V_t(0) \alpha}{2 F} \left[ 1 + \frac{1}{0.61 \rho \alpha} \left( \frac{1 - \exp(0.61 \rho \alpha)}{(0.61 \rho \alpha)^2} \right) \right]$$

(3.13)

where $\delta V_t(0)$ is the terminal velocity error at zero height. $F$ monotonically decreases from unity to zero at about an arc distance $\rho a = 3.3 / T$, a direct result of a height-dependent mass density. Thus, although error increases with height because of $\alpha^2$ in (3.13), this increase is offset by a decrease in $F$. For example, even when $\alpha$ is as large as 45° (0.79 rad) the error is $W_a$ is less than 0.2 of the error in $V_t$ at heights of 10–15 km. Because $W_s$ is well approximated by $W_a$ for $\alpha$ small, it is concluded that bias errors in $W_s$ are significantly less than those in $V_t$ and small, as will be seen in Section
employed to evaluate \( W_a \) uses four grid values different than the grid value at which \( W_a \) is evaluated.

**a. Vertical wind standard deviation \( \sigma_z \)**

To estimate vertical wind SD \( \sigma_z \) at various heights, for illustrative purpose we evaluated (2.3) and (2.10) for \( d=20 \text{ km} \), \( \Delta\rho=0.5 \text{ km} \), \( \Delta\alpha=0.5^\circ \), and assumed a vertical mass density variation \( \gamma/\gamma_0=\exp(-0.113z) \). Furthermore, we choose \( \sigma_x^2=\sigma_y^2 \) equal to a constant independent of grid-point location, a condition that results if a 4-point linear interpolation of range-averaged Doppler values is used (Appendix A) and for convenience \( \sigma_y=\sigma_z \). Equal interpolated variance \( \sigma_z^4=\sigma_x^4 \) also occurs when isotropic turbulence is the dominant variance-producing mechanism (instead of shear) and linear interpolation is applied to data that is not range-averaged. Combining (A4) with (3.7), we obtain

\[
\sigma_z^2=\sigma^2=\frac{0.019K_B\rho w\Delta V}{n_z\Delta\theta}.
\]

Even though \( \sigma_z^2(\bar{V}) \) increases with range, range averaging maintains \( \sigma^2 \) independent of \( r_z \). Fig. 2 shows the normalized SD \( \sigma_z/\sigma_1 \) as isopleths (for \( \sigma_1=1 \)) on constant height surfaces. Although we cannot normalize \( \sigma_z \) to \( d \), we have [for small \( \alpha \) where the second term in (2.10) is assumed negligible] \( \sigma_z=\sigma_x \), and [for fixed arc height and grid spacing] \( \sigma_z \) becomes proportional to \( (d/20)^4 \). Therefore, Fig. 2 isopleths multiplied by \( (d/20)^4 \) will give \( \sigma_z \) for other station separations and is the reason why \( x \) and \( y \) are normalized to \( d \) in this and following figures. Similarly, when \( \sigma_1 \) differs from 1, \( \sigma_z \) can be obtained by multiplying isopleth value by \( \sigma_1 \). Finally, \( \sigma_z \) can be deduced for small \( \alpha \) and grid or angle spacings different than that stipulated in Fig. 2 by multiplying it by \( 0.5/\Delta\rho \) or \( (\Delta\alpha/0.5)^4 \).

Assuming \( \sigma_1=\sigma_2=10^{-1} \text{ m s}^{-1} \) as a reasonable standard deviation for interpolated grid values, it is seen (Fig.2) that vertical velocity estimates have probably acceptable precision for altitudes up to 10 km. At \( z=14 \), \( y=60 \text{ km} \), the vertical velocity SD is 2.4 m s\(^{-1}\).

The precision could be improved by increasing 1) the number of radar echos processed per \( \bar{V} \) estimate (at the expense of longer scan time) and 2) the number of \( \bar{V} \)’s which are spatially averaged (at the expense of spatial resolution). The first procedure reduces \( \sigma_1 \) if errors are substantially due to finite spectrum width \( w_x \). Wind estimate variance, related to beam pointing errors (Section 3), can only be decreased by spatial averaging. Interpolated Doppler variance depends on \( \bar{V} \) sample density, mean Doppler variance \( \sigma_z^2(\bar{V}) \), and interpolation method (Appendix A).

**b. Standard deviation of horizontal wind magnitude and direction**

Since dual-Doppler derived horizontal wind most likely would be displayed on horizontal planes as wind
magnitude $W_H$ and direction $\psi$, it is preferable to have SD's of these quantities. Using the definitions

$$W_H = (W_z + W_y)^2, \quad \psi = \tan^{-1} \left( \frac{W_y}{W_z} \right),$$

(4.3)

and assuming errors concentrated about the expected values $\bar{W}_z, \bar{W}_y$, the magnitude variance $\sigma_M^2$ and direction variance $\sigma_\psi^2$ can be approximated by (Papoulis, 1965, p. 212),

$$\sigma_M^2 = \left( \frac{\partial W_H}{\partial W_z} \right)^2 \sigma_z^2 + \left( \frac{\partial W_H}{\partial W_y} \right)^2 \sigma_y^2 + 2 \frac{\partial W_H}{\partial W_z} \frac{\partial W_H}{\partial W_y} \mu_{zy},$$

(4.4)

with a similar formula for $\sigma_\psi^2$. The covariance $\mu_{zy}$ of $W_z$ and $W_y$ is defined as

$$\mu_{zy} = E[(W_z - \bar{W}_z)(W_y - \bar{W}_y)] = \sigma_z \sigma_y r,$$

(4.5)

where $r$ is the correlation coefficient of $W_z, W_y$. Expressing $W_z, W_y$ in terms of the independent variables $V_1, V_2, V_3, W_a$ and substituting into (4.5), we obtain

$$r = \frac{r_z^2(s-d) \sigma_z^2 + r_y^2(s+d) \sigma_y^2}{(2d)^2 \rho \sigma_z \sigma_y} \cos \alpha,$$

(4.6)

where $\sigma_z, \sigma_y$ are determined from (4.1). Solving (4.4) yields the magnitude and direction variance formulas

$$\sigma_M^2 = \sigma_z^2 \cos^2 \psi + \sigma_y^2 \sin^2 \psi + 2 \sigma_z \sigma_y \sin \psi \cos \psi,$$

(4.7a)

$$\sigma_\psi^2 = \sigma_z^2 \sin^2 \psi + \sigma_y^2 \cos^2 \psi - 2 \sigma_z \sigma_y \sin \psi \cos \psi.$$

(4.7b)

Comparing (4.7a) and (4.7b) we notice that normalized wind direction variance $W_H^2 \sigma_\psi^2$, at direction $\psi$, is obtained from $\sigma_\psi^2$ for wind direction $\psi + \pi/2$. Therefore we can obtain direction SD from magnitude SD.

In Fig. 3 we show the isopleths of normalized magnitude SD, $\sigma_M/\sigma_1$, for $\psi = 0$ (i.e., $\sigma_x/\sigma_1$) at $z = 0$ km. Since $\sigma_z$ is identical to $\sigma_\psi$, it can be shown from (2.3) that $\sigma_z$ is relatively independent of height and that the normalized SD is less than 2 everywhere over the $x, y$ domain indicated in the figure. Fig. 4 shows the normalized SD for wind perpendicular to the baseline. We see, as expected when both radar radials have directions close to that of the wind, that the smallest SD occurs in a region about the baseline bisector. Again, if the interpolated Doppler SD is less than $10^{-1}$ m s$^{-1}$, the SD in $y$-directed wind will be less than 1 m s$^{-1}$ up to 14 km for most of the area displayed.

Although $\sigma_M(\psi = 0)$ and $\sigma_M(\psi = \pi/2)$ are symmetrical about the plane $x = 0$, $\sigma_M$ is not symmetric in general because $\mu_{zy}$ has, for arbitrary $\psi$, odd symmetry about plane. For example $\sigma_M(\psi = 45^\circ)$ for $x$ negative has the same value as $\sigma_M(\psi = 135^\circ)$ for $x$ positive. Thus, $\sigma_M$ isopleths for $\psi = 45^\circ$, $\psi < 0$ are the mirror image of isopleths for $\psi > 0, \psi = 135^\circ$. 

**Fig. 3.** Isopleths of normalized standard deviation for horizontal wind magnitude $\sigma_z/\sigma_1$ at $z = 0$. Wind parallel to baseline ($\psi = 0$). 

**Fig. 4.** Normalized standard deviation for horizontal wind magnitude $\sigma_z/\sigma_1$. Wind perpendicular to baseline ($\psi = \pi/2$). $d = 20$ km.
Fig. 5. Polar plots of horizontal wind speed standard deviation $\sigma_H/\sigma_1$ at $z=2$ km, $x=0, 2d, y=2d, 3d$. Wind direction standard deviation is obtained by rotating each polar diagram by $90^\circ$.

Since the SD of wind magnitude and direction are functions of wind direction and speed, we have illustrated this dependence in Fig. 5 for nine selected points ($x=0, 20, 40; y=20, 40, 60$) at $z=2$ km. From the displays of $\sigma_H$ vs $\psi$, the normalized wind direction SD, $\psi_H/\psi_1$, can be obtained by rotating the figures $90^\circ$. Thus where magnitude SD is smallest, the normalized wind direction SD is largest.

This is seen in Fig. 6 which shows the relation between standard deviations of wind ($W$) and interpolated Doppler velocities ($V_1, V_2$). In Fig. 6 the expected values $V_1, V_2, W$ are identified as vectors and the Gaussian-shaped functions depict the scatter, proportional to $\sigma_1, \sigma_2$, about $V_1, V_2$. In case 1, where the expected wind is along the bisector of angle $\beta$, the projection of the distributions $P(V_1), P(V_2)$ along the direction $W$ is, for small $\beta$, accomplished without large increase in width along $W$. However in case 2, where $W$ is directed orthogonal to the bisector, we note that the distributions projected along $W$ have increased standard deviations. Although the magnitude SD of $W$ in case 1 is small, the uncertainty in $W$ direction is larger than in case 2 because there is a large angular region in case 1 over which the projected values of $V_1, V_2$ have intersections. It can be shown, neglecting $V_i$, that wind speed on constant $\alpha$ planes has minimum variance when the wind is directed along the bisector of the angle $\beta$. This fact is evident in the polar diagrams of Fig. 5. Finally, because of symmetry, it is deduced that the polar plots for negative $x$ are those in Fig. 5 rotated about the axis $\psi=90^\circ$.

An estimate of the errors in wind speed and direction can best be illustrated by example. Suppose that for $x=20$ km, $y=40$ km, $z=2$ km we estimate a dual-Doppler mean wind $W_H=10$ m s$^{-1}$ and mean direction $\psi=45^\circ$, and assume the interpolated SD $\sigma_1$ is about 1 m s$^{-1}$. Then $\sigma_\psi$ (Fig. 2) is 2.5 m s$^{-1}$, $\sigma_H$ (Fig. 5) is about 1 m s$^{-1}$, and the wind direction SD $\sigma_\psi$, obtained by rotating the polar diagram by $90^\circ$, is

$$\frac{1.8 \sigma_1}{W_H} \approx 0.18 \text{ rad},$$

that is, the wind direction SD is about $10^\circ$. Of course the smaller the wind speed the larger the error in wind direction.

5. Wind variance dependency on radar separation

In this section we treat the relation between wind variance and radar separation for $\alpha$ small, whereby $W_\alpha$ approximates $W_\sigma$, and $\sigma_\alpha$ at constant arc height $\rho\alpha$ estimates $\sigma_\psi$ at constant height $z$. We also assume $\sigma_1, \sigma_2$ independent of grid location. The factor multiplying $\sigma_1^2+\sigma_2^2$ [in Eq. (2.10)] is larger than unity.

Fig. 6. Wind magnitude standard derivation depicted as projections of intersections between $V_1, V_2$ normals on true wind vector $W$ line. Case "1" wind parallel to bisector of $\beta$ and case "2" orthogonal to bisector. Vectors lie on a constant $\alpha$ plane. $V_1, V_2$ and $W$ are vectors.
and if \( \sigma^2 \) is equal to or less than \( \sigma_1^2 + \sigma_2^2 \), we can neglect \( C_2 \).

Using \( d \) as a normalizing factor for the range distances \( r_1, r_2 \) and \( \rho \), the variance of the quasi-vertical wind \( W_z \) at the unnormalized height \( \rho \alpha \) is then

\[
2d\Delta\alpha \sigma^2'((r'_1 r'_2)^2) F_1(\rho \alpha) = \frac{\sigma^2(W_z)}{(2\Delta\rho)^2(2\rho)^2},
\]

where primes denote distances normalized to \( d \). Eq. (5.1) shows that, for fixed arc height, grid spacing and normalized location, the quasi-vertical wind variance \( \sigma^2(W_z) \) is linearly proportional to station separation.

However, grid spacings \( \Delta \rho, \Delta s \) and interpolation radius \( r_0 \) would probably be set proportional to the data spacing \( r_2 \Delta \theta \) (for \( s \geq 0 \)) as discussed in Section 2b of the Appendix. Similarly the angle increment \( \Delta \alpha \) would be proportional to \( r_0 \Delta \theta / \rho (s \geq 0) \). Therefore we assume

\[
\rho \Delta \alpha = \Delta \rho = \Delta s = C_3 \sigma \rho \Delta \theta,
\]

for \( s \geq 0 \), \( C_3 \geq 1 \),

where \( C_3 \) is a constant of proportionality that determines the number of data, spaced angularly at \( \Delta \theta \), used in interpolation (Appendix A, Section 2b). Substituting these and (A14) for \( 1 - R^2 \) into (3.7) to obtain \( \sigma_2^2, \sigma_3^2 \) and solving (5.1), we find

\[
\sigma^2(W_z) \approx \frac{0.01K_2\sigma_\rho \Delta V}{d n_C^3(\Delta \theta)^2} \left( \frac{r'_1}{r_0} \right)^3 \left( \frac{r'_2}{r_0} \right)^3 \left( \frac{r'_3}{r_0} \right)^3 \frac{F_1(\rho \alpha)}{(\rho \alpha)^2},
\]

\[
s \geq 0.
\]

We conclude that when shear is the dominant variance-producing mechanism and grid spacing is proportional to data spacing, \( \sigma^2(W_z) \) is, at fixed arc height and normalized grid location, inversely proportional to station separation, proportional to the beamwidth, and inversely proportional to the range sample density \( n_r \). The surprising result that \( \sigma^2(W_z) \) decreases as \( d \) increases, along with the fact that \( \sigma_2^2, \sigma_3^2 \) are independent of \( d \), implies that dual-Doppler coverage area can be increased to any desired size without increasing wind estimate variance. However, the resolution

\[
2d\Delta\rho = 2C_3\sigma \rho \Delta \theta
\]

of vertical velocity will be linearly proportional to \( d \) for constant normalized grid location. As an example, a \( 1^\circ \) beamwidth, range \( r'_1 = 2l \) and \( d = 20 \) km results in a 1 km resolution. As shown below, \( r'_2 = 2l = r'_1 \) is a location where \( \sigma^2(W_z) \) is a minimum. Furthermore when \( \rho \gg d \) and \( s \ll d \), \( \sigma^2(W_z) \) becomes proportional to \( \rho \) at fixed arc height and to the square of \( \rho \) for fixed \( \alpha \).

To estimate \( \sigma^2(W_z) \) assume that \( r'_1 = 2l = r'_2, K_s = 10^{-2} \) \( s^{-1} \), \( n_r = (200 \text{ m})^{-1} \), \( d = 20 \) km, \( \Delta \theta = \theta_0 = (1/60) \) rad, \( C_3 = 1, \Delta V = 1 \). For an arc height \( \rho \alpha = 10 \) km, \( F_1(\rho \alpha) \approx 4 \); thus

\[
\sigma^2(W_z) \approx 4 \text{ m}^2 \text{s}^{-1}.
\]

To assess an approximate spatial dependence of \( \sigma^2 \) on constant \( \alpha \) planes, we ignore the \( \rho \alpha \)-dependence in \( C_1 \) and assume \( \sigma_1^2, \sigma_2^2 \) are constant. Thus \( \sigma^2 \) is proportional to

\[
\sigma^2 \alpha^2 \alpha^2 \frac{r'_1 r'_2}{4d^2}.
\]

Along lines of constant radius \( R_s = \rho \alpha^2 \) we find that

\[
\sigma^2 \alpha^2 \frac{d^2 (R_s^2 - 1)}{4} + \rho^2.
\]

Therefore \( \sigma^2 \) is minimum along a circular arc passing through the radar sites, a line near which we should have the most precise \( W_z \) estimates. The angle subtended by \( r_1, r_2 \) is equal to \( \pi/2 \) along this line.

6. Application to an experiment (20 April 1974)

The derived formula can be used to estimate error in dual-Doppler deduced wind where error is assumed related to spectrum width. We have selected the dual-Doppler data and analysis presented in a companion paper (Ray et al., 1975) to compute wind SD derived from that experiment. For sake of simplicity we calculate horizontal wind speed errors at a height of 1 km where we can ignore vertical velocity variance. Contours of speed SD superimposed on a field of wind vectors are shown in Fig. 7. The SD is obtained from (4.7a) in which wind direction estimates are used for \( \beta \) and \( \sigma_x, \sigma_y \) are approximated by \( \sigma_x, \sigma_y \) [Eq. 2.3]. The interpolated Doppler velocity variances \( \sigma_x^2, \sigma_y^2 \) are computed using the distance-weighted interpolation technique described by Ray et al. This interpolation is similar to that described in the Appendix, the exception being that the influence radius on a horizontal plane is 1.5 km while along the vertical it is 1 km. Therefore \( \sigma_x^2, \sigma_y^2 \) are computed using (A7) and (3.6) to relate spectrum width \( \omega_x \) to interpolated Doppler estimate variance. Doppler width estimates, computed using

---

\( \sigma^2(W_z) \approx 4 \text{ m}^2 \text{s}^{-1} \)

---

\( \sigma_2^2(\rho \alpha) \approx 4 \text{ m}^2 \text{s}^{-1} \)

---

**Fig. 7.** Contours of wind speed standard deviation (m s\(^{-1}\)) for data collected at 1610 CST 20 April 1974. Wind vectors derived by Ray et al. (1975).

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*The factor of 2 is present because the finite-difference approximation used is centered differences.*
the pulse pair technique described by Berger and Groginsky (1973), were typically 5–6 m s\(^{-1}\) showing a slight increase with range.

The wind speed SD is relatively uniform between 0.2 and 0.4 m s\(^{-1}\) over a large region of the storm (Fig. 7). Higher SD (i.e., 0.6 m s\(^{-1}\)) in the northeast sector of the storm is a result of the wind being nearly perpendicular to the bisector of \(\bar{r}_1\), \(\bar{r}_2\). This location corresponds to \(y/d = 2\), \(s/d = 1\) (Fig. 5), where we see the ratio of speed SD for winds perpendicular and parallel to the bisector is about 2, a result supported by the error field of Fig. 7. Insofar as wind estimate errors are spectral-width-related, the error fields shown in Fig. 7 give the confidence one has in the dual-Doppler results published by Ray et al. (1975).

7. Summary and conclusions

The results of this work predict the variance of Cartesian wind components synthesized from dual-Doppler velocity estimates combined with the terminal velocity and the continuity equation. We give an example of the wind variance field and relate it to interpolated Doppler velocity variance for two interpolation schemes: a linear 4-point bivariate and a distance-weighted interpolation.

Vertical wind has variance larger than horizontal wind for most grid locations. Using an example it is concluded that vertical velocity can be estimated with a standard deviation less than a few meters per second up to heights near 14 km for extended regions of space (i.e., about \(4d \times 4d\) where \(2d\) is radar separation).

When the interpolation volume is matched to data spacing, it is shown that the dual-Doppler coverage area can be increased by increasing station separation without increasing wind estimate variance. However, increased coverage area is compromised by poorer resolution.

We show that horizontal speed estimate variance is a function of true wind direction, and that direction estimate variance is an inverse function of wind speed as well as a direct function of true direction. Best wind magnitude estimates are made when the true direction lies along the bisector of the angle between radius vectors drawn to the radars and the best direction estimate when the wind is perpendicular to this bisector. Results predict, for station separations of 40 km and altitudes below 6 km, minimum vertical velocity variance near the circular arc through the radar positions and, for higher altitude, near the radar's vertical.

Terminal velocity, determined from reflectivity estimates and an assumed dropsizes distribution, has more variance due to uncertainties in the size distribution than in reflectivity estimates. For typical dual-Doppler geometry, bias errors in vertical wind velocity due to particle fallspeed are significantly smaller than those in terminal velocity estimates for any reasonable heights (e.g., 0–20 km). Thus the selection of a \(V_{t,z}\) relation should not be critical. Finally an air density decrease with height significantly increases the vertical velocity estimate variance.

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APPENDIX

Variance Reduction for Interpolated Velocities

Doppler velocity data collected with equal spacings of radar range and beam position result in dual-Doppler radar fields having no data location coincidence. Therefore interpolation is required to locate data on a common grid. Although there are many different interpolation techniques, discussion is limited to two methods. A linear 4-point bivariate interpolation is applicable if data are acquired on constant \(\alpha\) planes. A second method concerns data weighted in proportion to the space-time distance between data and grid location.

If data are not acquired on planes of constant \(\alpha\), then interpolation must be four-dimensional (three space plus time). More than assigning Doppler values at grid locations, interpolation provides Doppler estimates \(V_1\), \(V_2\) that have variance smaller than that associated with \(\bar{V}\). We now determine variance reduction for each interpolation scheme so that with \(\sigma^2(\bar{V})\) as an input one may estimate the SD of dual-Doppler derived wind.

Although there are elegant techniques to determine weighting functions and a rigorous formula to estimate variance of interpolated values (Panchev, 1971), these solutions require spatial correlation functions of error and unknown wind fields. For simplicity, we assume a weighting function and present an estimate of variance reduction.

1. Linear interpolation on constant \(\alpha\) planes

Each grid point will be in a block bounded by four data points (two at constant range and two at constant beam position); thus the interpolated value is given by

\[
V_\alpha = V(\theta_0 + \Delta \theta, r_1 + q \Delta R) = (1 - p)(1 - q)V_{00} + p(1 - q)V_{10} + q(1 - p)V_{01} + pqV_{11},
\]

where \(V_{00}\) is the data value at \(\theta_0, r_1\), \(r_1\) and \(\theta_0\) define pulse volume position on an \(\alpha\) plane. \(V_{01}\) is the data point at constant range, but displaced \(\Delta \theta\), where \(\Delta \theta\), \(\Delta R\) are data-point spacings and \(p\) and \(q\) ranging from 0 to 1, define grid location within the data block.
Assuming \( \sigma^2(\bar{V}) \) equal around the block, the variance \( \sigma^2 \) at the grid point is

\[
\sigma^2(\bar{V}) = (1-p)^2(1-q^2)+p^2(1-q^2) + q^2(1-p^2)+p^2q^2 = F(q,p).
\]

(A2)

It has been shown (Miller and Strauch, 1974) that if \( p \) and \( q \) are independent and uniformly distributed over the block \( \Delta R \), then the expected variance is

\[
E[\sigma^2(\bar{V})] = \int_0^1 F(q,p)dpdq = 4/9 = 1 - R^2.
\]

(A3)

Thus the normalized variance reduction \( R^2 \) of interpolated velocity is \( 5/9 \) independent of grid location. If grid density is uniform and coarse, not all data may be used to reduce variance and resolution will be lost without benefiting variance reduction. On the other hand, if grid density is fine, interpolated values are correlated and the SD \( \sigma_x \) of the quasi-vertical wind derived earlier needs to be modified to account for this correlation. However no adjustment is required in this case for the variance in \( W_x, W_y \). Because data spacing increases with \( R \), it may be preferable to use a grid spacing \( \Delta \), that increases in proportion to the larger data spacing \( r_0 \Delta \theta \) for \( s \geq 0 \). Furthermore since radial sample density is usually more than azimuthal density, range averaging of samples over the interval \( r_0 \Delta \theta \) (and \( r_0 \Delta \) for \( \bar{V}_3 \)) would give a further variance reduction with a resolution consistent with beamwidth (i.e., assuming \( \Delta \theta \) proportional to \( \sigma_w \)). In this case,

\[
R^2 = 1 - \frac{4}{9n_r^2 \Delta \theta}.
\]

(A4)

where \( n_r \) is the range sampling density and we assume range-averaged data density equal to or larger than grid density.

2. Volume averages of Doppler estimates

a. Spherical volume

We now determine \( R^2 \) when data are averaged within a volume of constant radius. The interpolated velocity \( V_1 \) (or \( V_3 \)) is assumed to be a linear combination

\[
V_1 = \frac{\sum C_i \bar{V}_i}{\sum C_i}
\]

(A5)

of data values \( V_i \), where \( C_i \) is a weight determined by the assumed function

\[
C_i = \begin{cases} \frac{r_0^3 - r_i^3}{r_0^3 + r_i^3}, & 0 \leq r_i \leq r_0 \\ 0, & \text{otherwise} \end{cases}
\]

(A6)

where \( r_0 \) is the so-called influence radius, and \( r_i \) the grid
to data point distance. The interpolated velocity variance follows directly from (A3) and is

\[
\sigma^2 = \frac{\Sigma C_i \sigma^2(\bar{V}_i)}{(\Sigma C_i)^2},
\]

(A7)

where we assume equal data variance \( \sigma^2(\bar{V}_i) \) within the averaging volume. When data samples are uniformly dense within the volume, the above discrete sum formula can be replaced by the integral

\[
\frac{\sigma^2}{\sigma^2(Y)} = \frac{1}{3N} \int_0^1 C^2(X)dX \left/ \left[ \int_0^1 C^2(X)X^2dX \right] \right.
\]

(A8)

where \( X = r_i/r_0 \) and \( N \) is the total number of data points within the volume. Substituting (A6) into (A5) and carrying out the lengthy although elementary integrations we obtain normalized variance reduction

\[
R^2 = 1 - \frac{1.69}{N}.
\]

(A9)

For uniform data spacing \( \Delta d \), \( N \) is

\[
N = \frac{4\pi r_0^3}{3(\Delta d)^3}
\]

(A10)

which, when substituted into (A9), gives

\[
(1 - R^2)^4 = \frac{0.64}{(r_0/\Delta d)^4}.
\]

(A11)

Eq. (A11) is plotted in Fig. A1. Because (A11) is only

![Fig. A1. Multiplying factor to obtain grid estimate standard derivation from data standard derivation for distance-weighted interpolation, using spherical averaging volume and uniform data spacing. \( \Delta d \) is data spacing.](image-url)
applicable when \( N \) is large (i.e., \( r_0 \gg \Delta d \)) we need to determine \( R \) for small \( N \). In order to assess the range of \( r_0 \) for which (A11) is a good approximation we evaluated \( R \) using the exact formula (A7) for grid points on and midway between data points. These solutions (dashed lines in Fig. A1) show the approximate formula (A11) is good when \( r_0/\Delta d \geq 1.5 \).

The variance reduction \( R^2 \) obtained from (A11) is applicable to uniform distribution of data within \( r_0 \), a condition not usual in radar data acquisition schemes. Therefore if data samples are range averaged over the interval \( r_1 \Delta \theta \), then

\[
R^2 = 1 - \frac{0.41 (r_1 \Delta \theta)^2}{r_0^3 n_r}. \tag{A12}
\]

b. Cylindrical volume

An averaging volume which fits naturally into the cylindrical coordinate system is one bounded about a grid point \( \rho, s, \alpha \) with respective limits \( \pm \Delta \rho/2, \pm \Delta s/2, \pm \Delta \alpha/2 \). This "pie" shaped averaging volume gives the advantage that interpolated grid value errors are uncorrelated and hence the error analysis presented in this paper is applicable without correction. Furthermore this averaging volume may prove useful in recognizing data anomalies because only one grid value is affected by an anomalous data value. In any case, because angular data spacing increases with range it is most reasonable to assume an interpolation volume that accordingly increases. However the evaluation of \( R \) appears difficult. To make an \( R \) estimate we assume that the volume is nearly cubic (i.e., \( \rho \Delta \alpha = \Delta s = \Delta \rho \)) and use (A9) as an approximation for \( R \), where \( N \) is now the number of data contained in the averaging volume. If \( r_0 \) is set equal to the volume diagonal \( \sqrt{3} \rho_0 \Delta \alpha/2 \), and if data spacing \( \Delta d \) within the volume is uniform and equal to \( r_1 \Delta \theta \), the variance reduction \( R \) is

\[
R = 1 - \frac{1.1}{(r_0/\Delta d)^3}. \tag{A13}
\]

In this case \( R^2 \) is smaller than in (A11) for spherical volume because for given \( r_0 \) we have smaller \( N \) in the quasi-cubic volume. Substituting for \( r_0 \) and again assuming range sample averaging plus a grid spacing \( \Delta \rho = C s_2 \Delta \theta \) (for \( s \geq 0 \)), we obtain a variance reduction factor for \( V_1 \)

\[
R_1^2 = 1 - \frac{1.7 (r_1)^3}{C s n_r r_1 \Delta \theta / r_2}, \tag{A14a}
\]

and for \( V_2 \)

\[
R_2^2 = 1 - \frac{1.7}{C s n_r r_2 \Delta \theta}, \tag{A14b}
\]

where \( C \) is a constant, equal to or larger than unity, which determines grid spacing. \( R^2 \) for \( s \leq 0 \) is obtained by exchanging \( r_2 \) with \( r_1 \).

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